P-delta or *P*-theta analysis?

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ABSTRACT

In earthquake design of structures, engineers generally check that inelastically responding structure displacement demands are less than the displacement capacities. The displacement demands for structures with the same period, damping, and hysteretic behaviour, can be found. When so-called global " (P-Δ) analysis" (also referred to as "second order analysis", "higher order analysis", or "consideration of geometric nonlinearity") is considered as a correction to the "non *P-delta* analysis", the resulting displacement demands are not unique. This is because the effect of *P-delta* depends on the height to the centre of weight, L. While this concept is not new, this paper illustrates this concept for simple structures with (i) dynamic stability considerations, and (ii) with time history analysis, and (iii) discusses the implications on multistorey structures. It is shown that the lateral force in the hysteresis loop is decreased by "$P.\Delta/L$" or "$P.\theta$" where P is the weight above the level considered, and θ is equal to "Δ/L ". As a result, "*P-theta* (P-θ) analysis" may be a better term to describe this phenomenon than "*P-delta* (P-Δ) analysis". It is shown that the net post-elastic stiffness of a storey with a bilinear loop, $r_{net} = r - P/(kL)$, where k and r are the initial lateral stiffness and post-elastic stiffness factors respectively, should be significantly positive to ensure system dynamic stability and mitigate the likelihood of cumulative displacements in only one direction. Because of this, a greater r is generally required for shorter structures than for taller structures.

1 INTRODUCTION

In seismic design, the key parameters used for structure demand and capacity comparison are generally taken as the force (which is related to the accelerations), and the peak displacement. The likely maximum force capacity of many inelastically responding structures is generally estimated relatively easily from the hysteresis loop. Displacements may be estimated by inelastic dynamic time history analysis (IDTHA), but this approach is generally too complex for engineers conducting routine designs. Generally, simple displacement estimation methods are obtained based on empirical approximations to the IDTHA results for a structure with a certain period, damping, and hysteretic behaviour. It is possible to generate displacement estimates for a variety of structures based on period, damping, and hysteretic behaviour according to methods in many standards.

The displacement estimate of the response may, or may not, explicitly consider *P-delta* (also referred to as "second order analysis", "higher order analysis", or "consideration of geometric nonlinearity") effects. *P-delta* tends to increase the structural period and decrease structural dynamic stability. It is desirable that the net post-elastic stiffness of a structure considering *P-delta* effects should be significantly greater than zero in order to mitigate the possibility of seismic ratchetting. *P-delta* may have different effects on structures with the same period, damping and hysteretic behaviour, so consideration of *P-delta* does not have a unique effect and this is not always appreciated.

There is a need to address this issue of the non-unique effect of *P-delta* so that it may be considered appropriately in design.

This paper seeks to address this need by seeking answers to the following questions:

1. For a simple structure with a certain, period, damping and hysteresis curve, but with different height to the structure's centre of mass, L, how do *P-delta* considerations affect the pushover response?

2. For the same structures with a certain lateral force reduction factor for a particular earthquake record, how do *P-delta* considerations affect the time history response?

3. For the same structure, how do *P-theta* considerations affect the response?

4. What are the implications for design of the findings above?

2 LITERATURE REVIEW

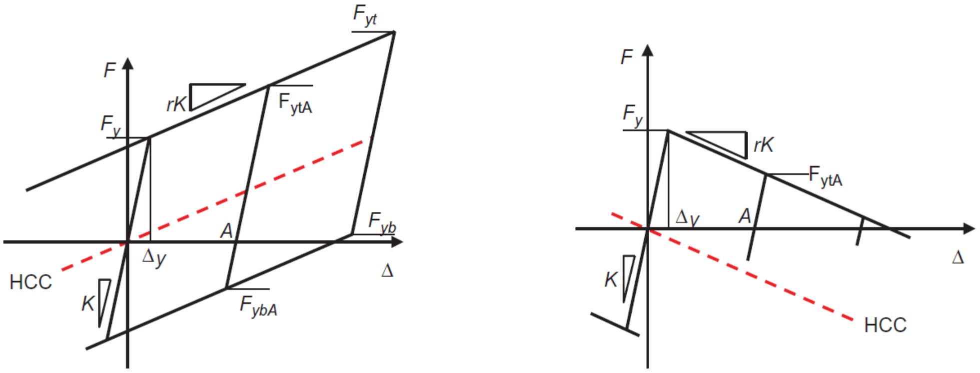
2.1 Dynamic stability

Figure 1 shows two different bilinear hysteresis loops (MacRae, 1994). The first has a positive post-elastic stiffness (Figure 1a), and another with a negative post-elastic stiffness (Figure 1b) which may be a result of material characteristics, buckling of the member, or *P-delta* effects. If a structure has a positive post-elastic stiffness and is oscillating about point A (on Figure 1a), the structure would have a lower yield strength in the negative direction resulting in a greater tendency for the building to yield back towards the zero-displacement position. Furthermore, when a structure is at the peak displacement, the velocity there is zero and it has significant potential energy. When the structure oscillates freely from this point, the displacement reduces and potential energy is transformed into kinetic energy and there will be significant velocity when it reaches the zero force line causing it to likely yield towards its initial point ($\Delta = 0$). Such a hysteresis loop is "dynamically stable". In contrast, a structure with a negative post-elastic stiffness, such as that in Figure 1b, when oscillating about point A, would be more likely to yield away from the zero-displacement position. Such a structure is "dynamically unstable" and the increased displacements in one direction are known as seismic ratcheting, and larger residual/permanent displacements after a major shaking event. A hysteresis centre curve (HCC) line can also be established. It has a strength $F_{HCC,i}$ given by Equation 1, where $F_{yt,i}$ and $F_{yb,i}$ are the yield forces corresponding to the interception of different elastic load/unload line, i, with the top and bottom yield curves, respectively. It is shown by the dashed lines in Figure. 1. Dynamic stability occurs when the secant stiffness from the origin to a point on the Hysteretic Centre Curve (HCC) is positive, which is when the HCC is in the first and third quadrants.

$$F_{HCC,i} = (F_{yt,i} + F_{yb,i})/2 \quad (1)$$

2.2 *P-delta* Effect on Hysteresis loops

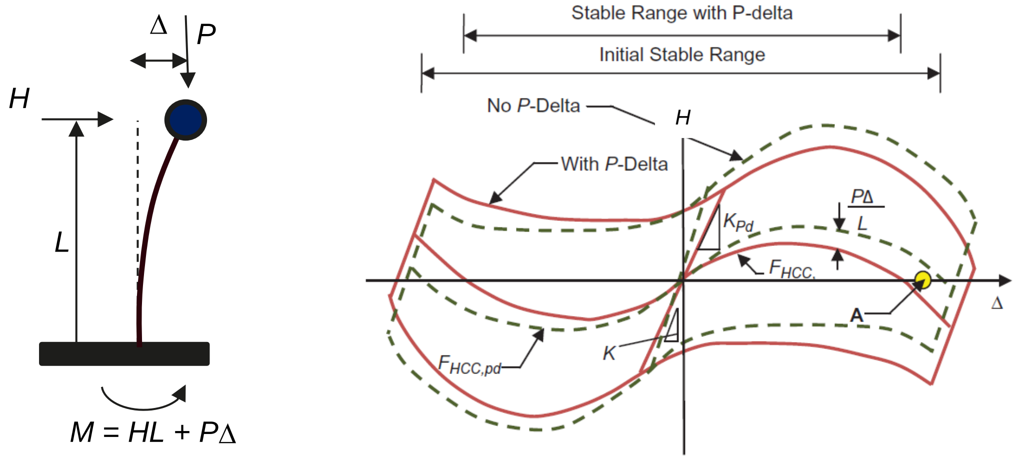
A vertical force, P, moving through a lateral displacement, Δ, causes an additional moment "$P\Delta$" at the base of a structure as shown in Figure 2a, where H is the lateral force resistance, and L is the height to the centre of mass of the structure. Moment equilibrium about the base of a cantilever column can be used to show a decrease in lateral force resistance due to the *P-delta* effect. Here, $H = H_o - P\Delta/L$, where H_o is the lateral force resistance if $P = 0$.



(a): Positive Bilinear Hysteresis Loop *(b): Negative Bilinear Hysteresis Loop*

Figure 1:Dynamic Stability (MacRae, 1994)

The effect of *P-delta* on a general hysteresis loop is shown in Figure 2b. The loop shown by the dashed line does not consider *P-delta* effects. The “stable range” in the figure is defined as the segment of the hysteresis loop in which the secant stiffness from the origin to a point on the Hysteretic Centre Curve (HCC) is positive. The second loop, shown by the solid lines, takes *P-delta* effects into consideration. At point A, yielding is more likely to occur in the negative direction when *P-delta* effects are ignored since the force required to yield in the negative direction is smaller than that in the positive direction. When considering *P-delta* effects, the structure is more likely to yield in the positive direction as the force required to yield in the positive direction is smaller than that in the negative direction.



(a) Structure *(b) Hysteresis Loop*

Figure 2: P-delta effect (MacRae 1994, Yeow et al., 2013)

2.3 Post-Elastic Stiffness of Bilinear SDOF Structures

For bilinear single-degree-of-freedom (SDOF) structures, the secant stiffness to the HCC is constant, and it has the same slope as the post-elastic stiffness of the oscillator. For these structures, dynamic stability is therefore obtained when the net post-elastic stiffness k_{net} (= $r_{net}k$) considering *P-delta* is greater than zero, where k is the initial stiffness without considering P-delta, and r_{net} is the post-elastic stiffness factor considering P-delta.

$$k_{net} = H/\Delta \qquad (2)$$

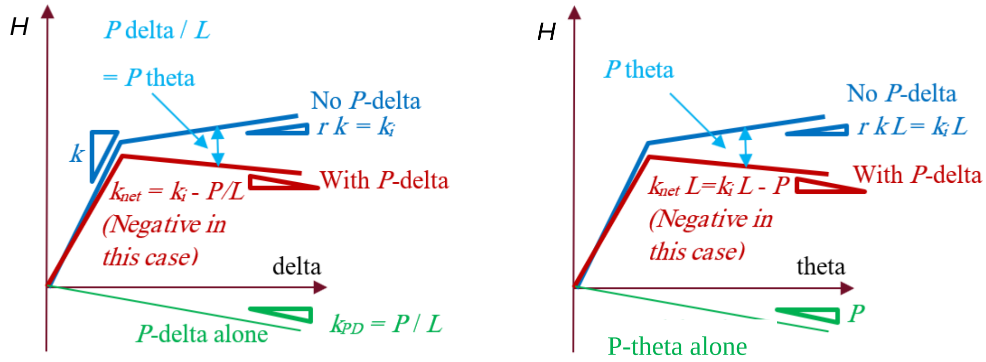
$$= (H_o - P\Delta/L)/\Delta \qquad (3)$$

$= k - P/L$ (4)

For $k_{net} > 0$ in the yielding range (where k is equal to the post-elastic stiffness before *P-delta* is considered, k_i), it is necessary that:

$$k_i > P/L \quad (5)$$

The effect of *P-delta* on post-elastic stiffness in terms of "force versus delta" is shown in Figure 3a and "force versus theta" in Figure 3b. It may be seen that the height to the centre of mass, *L*, is likely to influence the response.



(a): P vs delta *(b): P vs theta*

Figure 3: Force Versus Displacement Curve

3 METHODOLOGY

Single-degree-of-freedom (SDOF) numerical cantilever models were considered with a natural period, *T*, of 1.0s, axial load, *P*, of 10 MN. The cantilever considered an elastic beam-column element and a plastic hinge at element base. The hinge used the Ibarra-Krawinkler deterioration model to obtain elastic perfectly plastic (EPP) hysteresis behaviour. Six different heights to the centre of mass, *L*, used were 1m, 4m, 7m, 10m, 30m, and 1000m. For all structures to have obtain the same period, *T*, without the *P-delta* effect, the section flexural stiffness, *EI*, was computed in Equation 6, where m is the structural mass (= *P*/*g*), and *g* is the acceleration due to gravity. The associated lateral stiffness, k ($=3EI/L^3 = 4\pi^2 m/T^2$), is 4024 kN/m. For the inelastic analysis, the peak elastic force demand without *P*-delta, H_e, was divided by a lateral force reduction factor, *R*, of 3.0, to obtain the lateral yield force, H_y. At this strength, the yield displacement, D_y, is 20mm.

$$EI = 4\pi^2 m L^3/(3T^2) \quad (6)$$

For time history analysis, an initial stiffness proportional Rayleigh model is used with 5% damping at periods of 1.0s and 0.047s. The ground motion used is given in Table 1. Pushover analysis and time history analysis, with and without P-delta, are all performed using OpenSEES software (Mazzoni et al., 2006).

Table 1: Earthquake Record

Event:	Helena Montana - 01
Duration (s):	50
Date:	10/31/1935
Location:	Carroll College
Horizontal Direction:	HELENA.A_A-HMC180.AT2

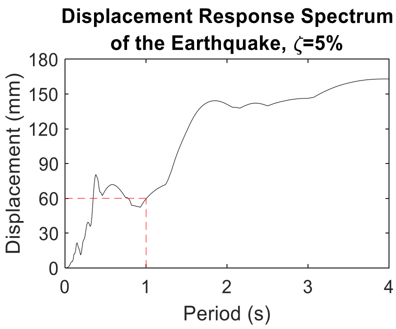
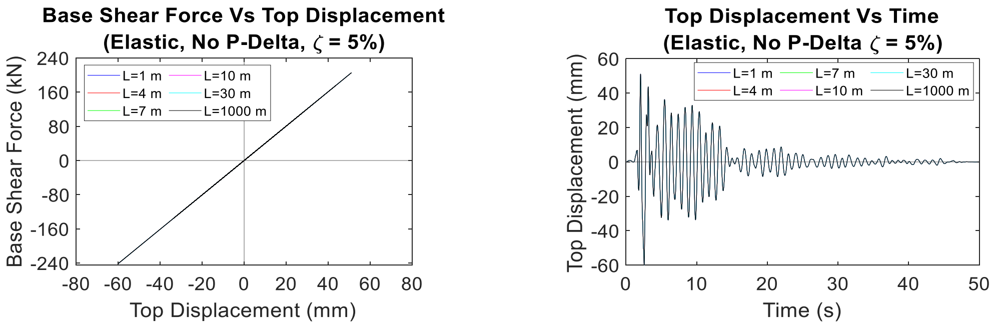


Figure 4: Record Displacement Response Spectrum (ζ = 5%)

4 BEHAVIOUR

4.1 Elastic Structure with No P-delta

Figure 4a shows that the elastic SDOF dynamic response to the earthquake record. It reaches a peak displacement and base shear force of 60 mm and 241 kN respectively (in the negative direction). The displacement is consistent with the response spectrum in Figure 4. The stiffness of 4024 kN/m is consistent with that computed above. Figure 5b confirms that all the elastic SDOF structures without *P-delta* effects, have the same time history response under the same earthquake record. This is as expected because the period and damping of all structures are identical.

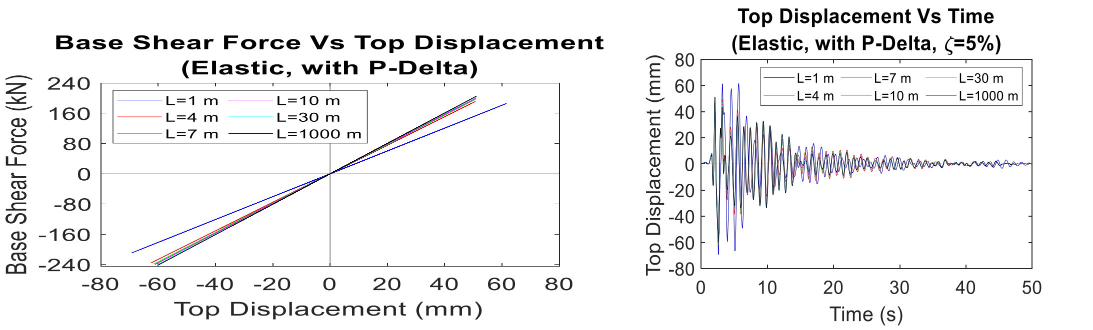


(a) Base Shear Force Versus Top Displacement *(b) Top Displacement Versus Time*

Figure 5: Elastic Time History Behaviour (No P-Delta)

4.2 Elastic Structure with P-delta

When *P-delta* is considered, the force displacement curve for the record excitation changes as shown in Figure 6a and key values are given in Table 2. While *P-delta* is applied to structures with the same response without *P*-delta, due to them having the same period, damping and hysteresis loop, the effect of *P-delta* is not unique, but it depends on *L*, and this causes different responses. The changes are because the hysteresis loop changes as shown in Figure 3, so the stiffness, *k*, is reduced by P/L according to Equation 4, and hence the period, T ($=2\pi/\sqrt{(m/k)}$), is increased. However, for very high *L*, the term P/L is small and the stiffness does not noticeably change so the response is similar to that with no *P*-delta. Only the structures with very low *L* are significantly affected by *P*-delta. Figure 6b shows that there is an effect on the top-displacement versus time plot. It may be seen here, and in Table 2, that the period of the short structure (*L*=1m) is increased, as is the displacement. This is expected when the spectral displacement increases with period as per Figure 4. Also, Table 2 shows shorter structure force demands decrease may be computed from the acceleration response spectra (not shown).



(a) Base Shear Force Versus Top Displacement *(b) Top Displacement Versus Time*

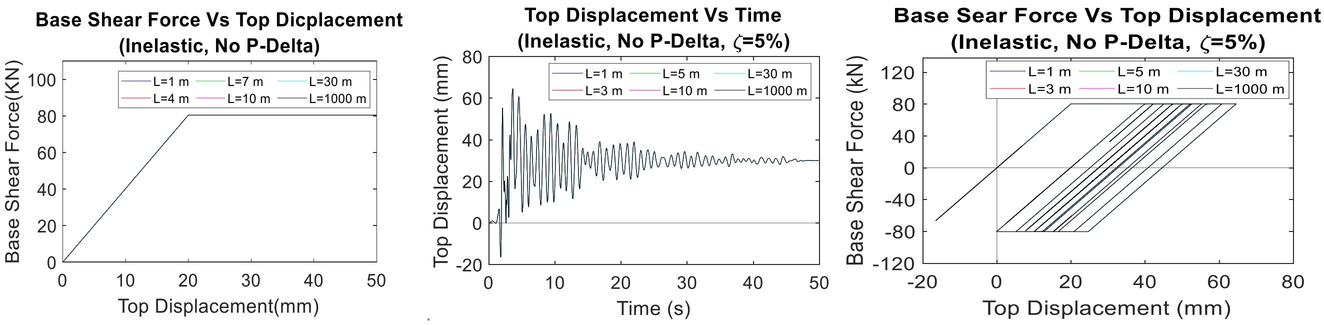
Figure 6: Elastic Time History Behaviour (With P-Delta)

Table 2: Time History Response Summary for Elastic Structures Considering P-Delta

Height, *L* (m)	**1**	**4**	**7**	**10**	**30**	**1000**
Maximum Top Displacement, Δ_m (mm)	69.2	62.5	61.4	61.0	60.3	60.0
Maximum Base Shear Force, V_m (kN)	209	236	238	239	241	241
$k_{net}\ (kN/m) = k_i - P/L$	3024	3774	3881	3924	3990	4023
T (s)	1.15	1.03	1.018	1.012	1.004	1.0001

4.3 Inelastic Structure with No P-delta

The lateral yield strength provided for the structure, H_y, is H_e/R = 241 kN/3 = 80.5kN, from Figure 5a. This is consistent with Figure 7a. Figures 7b and 7c show the inelastic time history response which is identical for all structures because their period, damping, and hysteresis loop are the same. Also, the peak displacement of 64mm shown in the figure is similar to that of the elastic structure of 60mm.



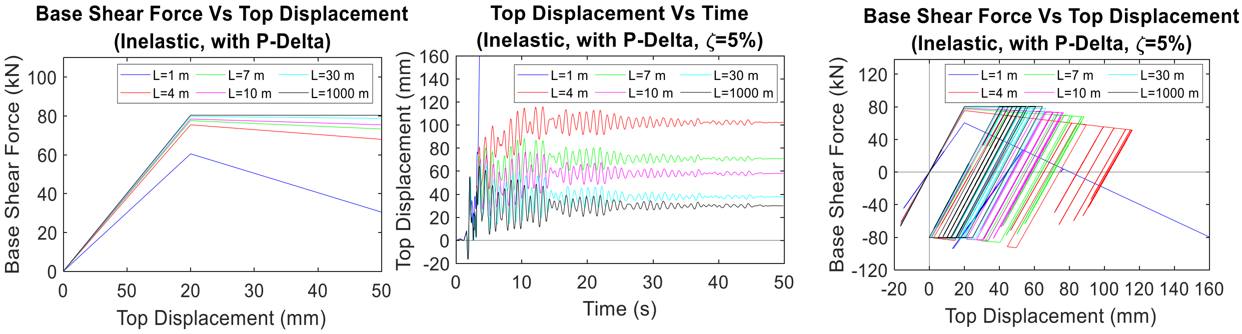
(a) Pushover *(b) Top Displacement Versus Time* *(c) Hysteresis Behaviour*

Figure 7. Inelastic Structure with No-P-delta

4.4 Inelastic Structure with P-delta

Figure 8a, like Figure 6a, shows that the term “-$P\Delta/L$” or “-$P\theta$” depends on the height, L. For the structure with L = 1m, this term causes a significant reduction in strength with displacement. Extrapolation of this line indicates that it will reach zero force at a displacement of about 80mm. Beyond this displacement it is no longer statically stable. That is, even without any ground shaking it will overturn. Also, by comparing with Figure 7a it can be seen that for very tall structures, for example, L = 1000 m, *P-delta* effects are negligible.

Figures 8b and 8c and Table 3 indicate hysteresis, as well as time history, behaviour. The reduction in strength from 80kN, which occurs at the yield displacement, Δ_y, is shown in Table 3 as "-$P\Delta/L$" for $\Delta = \Delta_y$ = 20 mm. For the shortest structure, with L = 1m, the reduction in strength at this displacement is 20kN/80kN = 25%. The net post-elastic stiffness ratio, r_{net}, considering *P-delta* becomes as low as -0.33 for the same structure. The shortest structure, with L = 1m, becomes statically unstable during the analysis and collapses so no peak or residual displacement are reported. In general, the shorter structures with the same *P-delta* effect have significantly greater peak and residual displacements. For the very tall structure, the response is similar to that with no *P-delta* effect.



(a) Pushover *(b) Top Displacement Versus Time* *(c) Hysteresis Behaviour*

Figure 8. Inelastic Structure with same P-delta

Table 3: Time History Response Summary for Inelastic Bilinear Structures Considering P-Delta

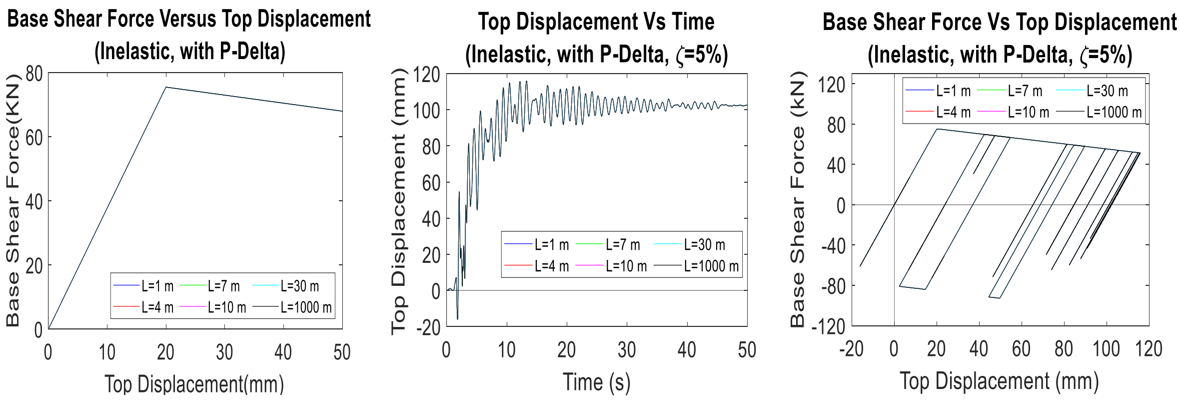
Height (m)	**1**	**4**	**7**	**10**	**30**	**1000**
-$P\Delta/L$ (kN) at Δ=20 mm	20	5	2.9	2	0.7	0.02
r_{net}	-0.33	-0.07	-0.036	-0.025	-0.008	-0.0002
Maximum Displacement, Δ_m (mm)	N/A	115.9	88.3	76.6	66.4	64.6
Residual Displacement, Δ_r (mm)	N/A	102.4	70.9	58.1	37.7	30.1

4.5 Consideration of *P*-theta

Based on the above discussion, structures have similar response before the *P-delta* effect is considered, may have significantly different response when the same *P-delta* effect is applied, depending on the height of the structure. The *P-delta* effect does not relate to a unique response for structures with the same period, damping and hysteresis loop shape. This is because the response depends on the height to the centre of mass. If, instead of considering *P*-delta, the parameter *P-delta*/*L* is considered, this may result in the same response for all structures because the reduction in the hysteresis loop is shown in Figure 3 to be $P\Delta/L$, which is also equivalent to $P\theta$.

To analyse structures with the same $P\Delta/L$, the ratio P/L must be made constant. For structures of different height, this can only be achieved by artificially modifying P. This is done so that $P = P_{standard} \times L/L_{standard}$ where standard values are selected for P and L, which are taken as mg, and 4.0m respectively. The mass, m, is kept the same for all structures, but the axial load P is artificially modified in each case so that the term "P/L" is constant for all structures. This artificial modification of P can be considered to be due to structures of different heights being placed on different planets, where the gravity force changes.

Figures 9a-c indicate that despite the different heights, all structures have the same response before global second order effects are considered because they have the same *P-theta* effect. As a result of the mechanics associated with force acting through the displacement, it is clear that the response is dependent not on $P\Delta$, but on $P\Delta/L$ which is equal to $P\theta$. It therefore makes sense to refer to this second order effect as the *P-theta* effect, rather than the *P-delta* effect.



(a) Pushover *(b) Top Displacement Versus Time* *(c) Hysteresis Behaviour*

Figure 9. Inelastic Structure with same P-theta

5 IMPLICATIONS

Implications of the findings from this paper are:

(i) Global second order effects on the inelastic seismic response are not defined only by P and Δ. Instead, they are also affected by the height to the centre of mass, L. Rather than the structure displacement, the structural drift ratio is more sensitive to these effects.

(ii) Because taller structures which deform predominantly linearly over the height with a certain displacement at the centre of mass have lower drifts than shorter structures with the same displacement, they are less sensitive to second order effects.

(iii) While structure of taller height that deform linearly over their height may be less susceptible to *P-theta* effects than equivalent shorter structures, other criteria may be more significant in making final design decisions. For example, structures of different heights may have different strengths. Also, taller structures are likely more susceptible to wind loading than are shorter structures. Also, the possibility of a moderate wind blowing on the structure at the time of earthquake shaking may tend to increase the likelihood of the structure ratchetting on one direction (MacRae et al., 2023).

(iv) While the height to the centre of mass, L, is important, structures with drift concentrations over their height are likely to be more susceptible to the *P-theta* effects. These may be mitigated by providing a system which prevents large drift concentrations in a storey. One way of doing this is by means of providing continuity of stiffness over the building height.

This has been described as the continuous column effect (MacRae et al., 2004) and (MacRae, 2011), and has more recently been described as the building spine effect, stiffback, or strongback.

(v) Dynamic instability may be avoided by providing structures with greater initial post-elastic stiffness before global second order effects are considered. Structures with drift concentrations in the lower stories, over short vertical heights, *L*, are most susceptible to increased displacements. Such structures include some base-isolated structures, where the height of the dissipator may be less than 300mm. To avoid large cumulative yielding displacements in one direction and to ensure dynamic stability, the post-elastic stiffness before consideration of global second order effects should be high (i.e. > P/L) to ensure that the net post-elastic stiffness is significantly greater than zero. The effect is less extreme in structures where one storey (with *L* greater than that of a base isolation device) is used as to isolate the structure as per the works of Hariri & Tremblay (2023), Tremblay & Darwiche (2023), and Tremblay (2023), but checks are still required.

(vi) Greater ratchetting displacements are expected for structures with greater drift limits, so decreasing the drift limit also is likey to result in better performance.

6 CONCLUSIONS

This paper describes the effects of higher order geometric effects of the overall behaviour of single degree of freedom systems subject to earthquake shaking. It is shown that:

1. For simple single-degree-of-freedom cantilever structures with the same period, damping, strength, strength and an elastic-perfectly plastic hysteretic characteristic, the pushover curve response without considering *P-delta* effects was the same. However, when second order effects were considered on the overall member, the structures with the smallest height to the centre of mass, *L*, had the greatest strength reduction. This strength reduction was $P\Delta/L$, and the stiffness reduction was P/L, confirming previous studies on the *P-delta* effect. The fact that the same *P-delta* caused different behaviour depending on the frame height, *L*, indicates that the term "*P-delta*" may not be the best term to describe this effect.
2. When these structures were subjected to a particular earthquake record, and the lateral force reduction factor, *R*, was 3, the shortest structure collapsed, while the tallest structure showed the same response as a structure with no *P-delta* considerations. Generally, shorter structures showed greater peak and residual displacements.
3. By keeping P/L constant, the inelastic response considering second order effects was the same for structures irrespective of the *P-delta* effect. This indicates that "*P-theta*" is a better term to describe this effect than "*P-delta*", and the effect on the hysteresis curve was dependent on the storey drift angle, theta (θ) rather than the displacement (Δ).
4. Implications of the above for design are that as the interstorey drift angle, θ, increases, the likelihood of ratchetting due to *P-delta* increases. The amount of ratchetting is likely to be significantly greater with greater θ, especially when the net post-elastic stiffness of the structure becomes negative. This may be mitigated by (a) making the structure stronger (and hence reducing the lateral force reduction factor) for structures with a fat or elastoplastic type hysteresis loop, (b) reducing the interstorey drift angle, θ, by using stiff members such as continuous columns, frames or walls, over the structural height which also take advantage of the centre of mass being high in the structure, (c) by reducing the interstorey drift angle, θ, by using a lower design drift limit, or (d) providing the structure a different hysteresis loop shape with a higher effective post-elastic stiffness at each level to counter the effect of P/L at each storey to obtain dynamic stability. This initial post-elastic stiffness needs to be greater for short elements/stories (such as base isolation devices) with large drifts, because P/L is high for these.

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